# Understanding and Implementing the Weight(ed) Overlap Add Fast Fourier Transform (WOLA FFT)

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Output channel characteristics of the rectangular-windowed Fast Fourier Transform (FFT) are not satisfactory for many applications. Figure 1 (below) shows the frequency response of two adjacent channels of a FFT (of arbitrary length) which exhibit significant main lobe overlap and high sidelobes. The high sidelobes skirt into neighboring channels resulting in spectral leakage.

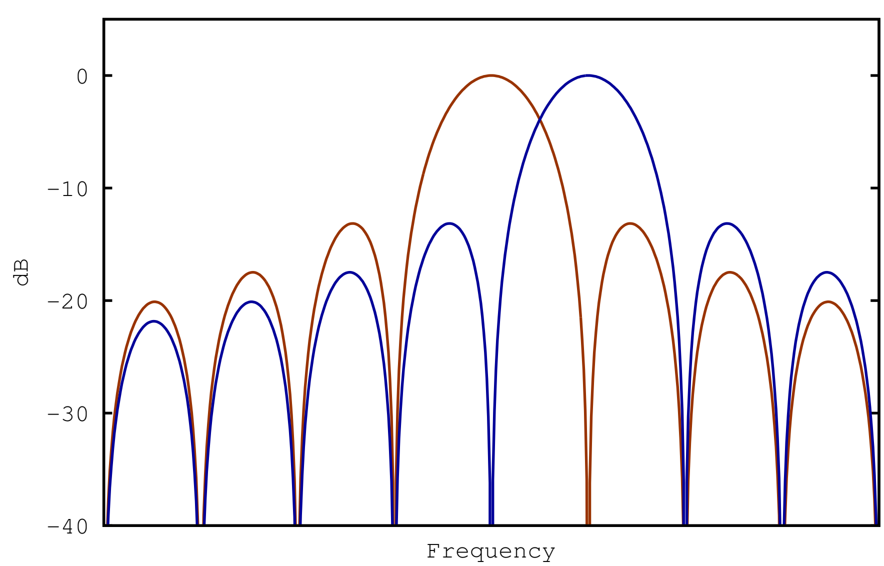


Figure 1: adjacent FFT channels response

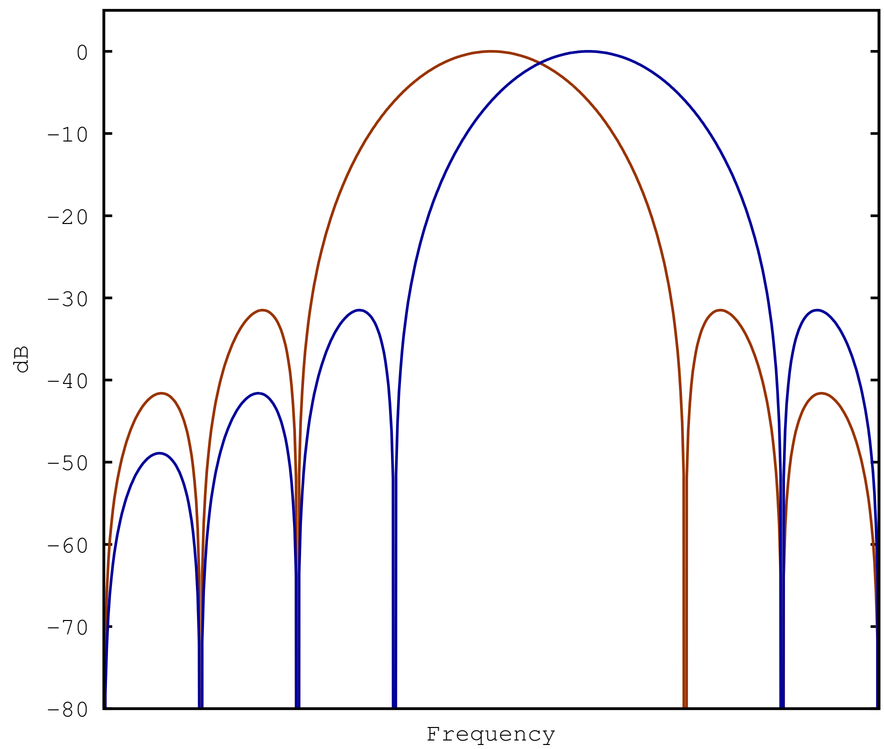


Figure 2: adjacent FFT channels Hanning response

The high sidelobes can be suppressed by weighting the FFT input vector. Figure 2 (above) shows the effect of weighting the FFT input vector with a Hanning cosine window: $x(n) \cdot 0.5(1 - \cos((2\pi n)/N))$. The first sidelobes are suppressed to $-31.6$ dB, which is a significant improvement over the $-13.2$ dB peak sidelobes shown in Figure 1 for the rectangular-windowed FFT; however, the main lobes have widened such that they overlap considerably. The two adjacent FFT output channels shown do not practically distinguish between frequencies that fall in the range where the main lobes overlap.

The FFT is an efficient $N \cdot log_2(N)$ time complexity algorithm that generates the same output, for a given input vector, as the less efficient $N^2$ time complexity Discrete Fourier Transform (DFT). The DFT is considered in the following paragraphs because it reveals its complex analysis sinusoids directly; however, the FFT is substituted for the DFT in practice for most applications.

The top graph of Figure 3 (next page) shows the frequency response for each of ten cosine-windowed (Hanning) adjacent channels representing a subset of the channels output from a DFT five times larger (i.e., five times longer input vector and five times as many output channels) than the transforms in Figures 1 and 2. The relative scaled frequency response of each of the ten channels is the same as the frequency response of the channels shown in Figure 2; however, more sidelobes are

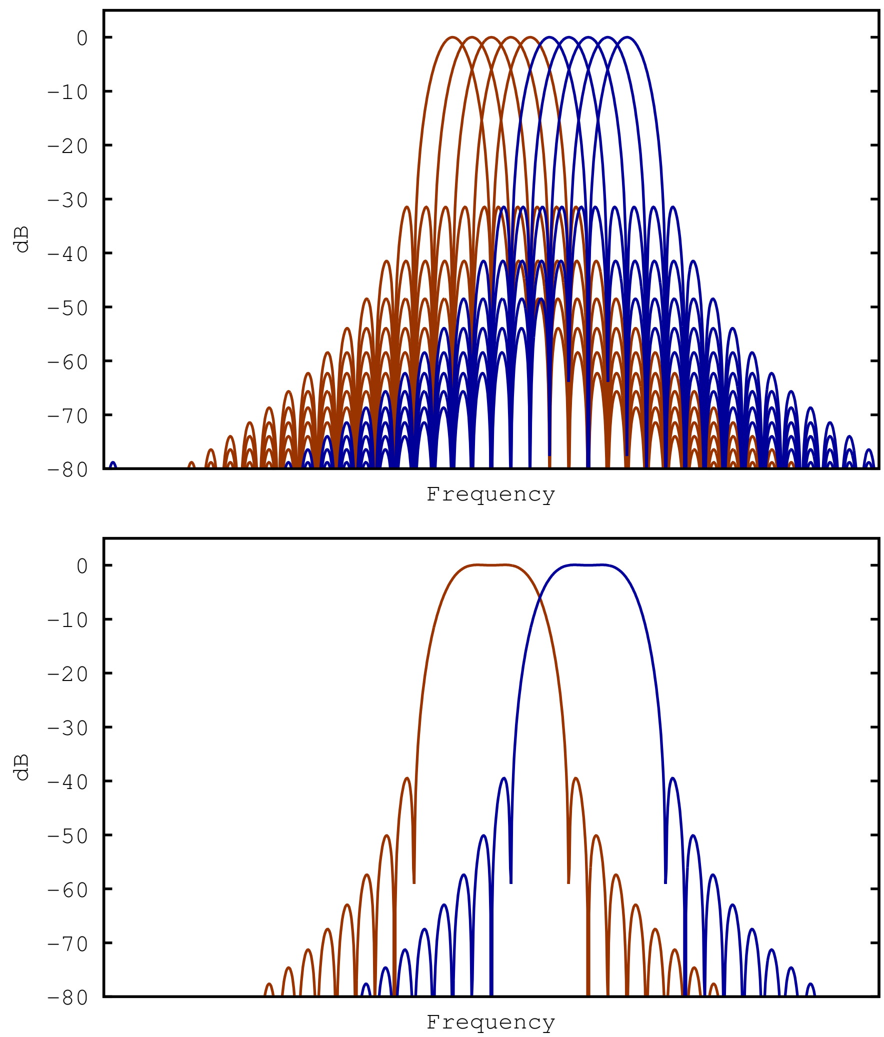


Figure 3: adjacent channels WOLA response

visible in the skirts in Figure 3. The bottom graph of Figure 3 shows the frequency response of two adjacent channels separated by the same center frequency span as the channels shown in Figures 1 and 2. The left channel is comprised of the complex sum of the left five channels of the graph above it and the right channel is similarly comprised of the complex sum of the right five channels of the graph above it.[1] The channels resulting from the summation exhibit flatter central passband response, increased sidelobe suppression, narrower sidelobes relative to main lobe width, and good channel separation (reduced main lobe overlap). If more than five channels were summed from the output channels of an even larger transform, the characteristics of the resulting summed output channels would continue to improve.

[1]The DFT complex analysis sinusoids are shifted such that zero phase coincides in the middle of the time series window rather than at the edges.

The cost of obtaining output channels with the desirable characteristics shown in the bottom graph of Figure 3 is apparently the generation of output channels to be summed from a transform much larger than the number of channels finally required; however, it can be shown with some algebraic manipulation that the same desirable channel characteristics can be obtained more efficiently by *weighting* a large input vector with a large real coefficient window which is subsequently *overlapped*, *added*, and then input into a smaller transform (e.g., *FFT*) of only the size necessary for the final number of channels required (i.e., WOLA FFT).

To present an example, channel one of an eight-channel DFT with the characteristics shown in the bottom graph of Figure 3 will be generated starting with the complex sum of channels three through seven of a larger forty-channel DFT. Through algebraic manipulation the need for the forty-channel DFT will be eliminated. Channels three through seven are chosen to be combined in the forty-channel DFT because the central channel, channel five, has the same digital frequency as channel one of the eight-channel DFT. The choice of an eight-channel DFT derived from a five times larger forty-channel DFT for the following example is somewhat arbitrary; however, it was constrained by the desire for it to be large enough to be interesting but not so large that the subsequent algebra is overly tedious.

Sum channels three through seven of a windowed forty-channel DFT:

$$\sum_{n=0}^{39} x(n)h(n)e^{-\frac{2\pi i}{40}3n} + \sum_{n=0}^{39} x(n)h(n)e^{-\frac{2\pi i}{40}4n} + \sum_{n=0}^{39} x(n)h(n)e^{-\frac{2\pi i}{40}5n} + \sum_{n=0}^{39} x(n)h(n)e^{-\frac{2\pi i}{40}6n} + \sum_{n=0}^{39} x(n)h(n)e^{-\frac{2\pi i}{40}7n}$$

Factor out the center channel five complex analysis sinusoid. The remaining terms within the parenthesis are the inner product of the input vector $\bar{x}$ times the residual WOLA window (the residual WOLA window terms are the same for all channel groupings 38-2, 3-7, 8-12, 13-17, 18-22, 23-27, 28-32, and 33-37, with their corresponding center channel factored out):

$$\sum_{n=0}^{39} e^{-\frac{2\pi i}{40}5n} \Big( x(n)h(n)e^{-\frac{2\pi i}{40}(-2n)} + x(n)h(n)e^{-\frac{2\pi i}{40}(-1n)} + x(n)h(n)e^{-\frac{2\pi i}{40}(0n)} + x(n)h(n)e^{-\frac{2\pi i}{40}(1n)} + x(n)h(n)e^{-\frac{2\pi i}{40}(2n)} \Big)$$

The channel five complex analysis sinusoid cycles five times (and other center channels 10, 15, 20, 25, 30, and 35, cycle a multiple of five times); therefore, the analysis terms can be grouped into an eight-channel DFT:

$$\begin{aligned}
&e^{-\frac{2\pi i}{8}0} \sum_{n=0,8,16,24,32} x(n)w(n) \quad + \quad e^{-\frac{2\pi i}{8}1} \sum_{n=1,9,17,25,33} x(n)w(n) \quad + \\
&e^{-\frac{2\pi i}{8}2} \sum_{n=2,10,18,26,34} x(n)w(n) \quad + \quad e^{-\frac{2\pi i}{8}3} \sum_{n=3,11,19,27,35} x(n)w(n) \quad + \\
&e^{-\frac{2\pi i}{8}4} \sum_{n=4,12,20,28,36} x(n)w(n) \quad + \quad e^{-\frac{2\pi i}{8}5} \sum_{n=5,13,21,29,37} x(n)w(n) \quad + \\
&e^{-\frac{2\pi i}{8}6} \sum_{n=6,14,22,30,38} x(n)w(n) \quad + \quad e^{-\frac{2\pi i}{8}7} \sum_{n=7,15,23,31,39} x(n)w(n)
\end{aligned}$$

$$w(n) = h(n) \Big( e^{-\frac{2\pi i}{40}(-2(n-19.5))} + e^{-\frac{2\pi i}{40}(-1(n-19.5))} + e^{-\frac{2\pi i}{40}(0(n-19.5))} + e^{-\frac{2\pi i}{40}(1(n-19.5))} + e^{-\frac{2\pi i}{40}(2(n-19.5))} \Big)$$

The next few lines of text and equations are a recipe for fully implementing the example eight-channel, five-overlap WOLA FFT, which was derived for one output channel on the previous page:

1) initially generate the WOLA window:

$$N = \text{output channels} \cdot \text{overlaps} = 8 \cdot 5 = 40$$

$$h(n) = 0.5(1 - \cos((2\pi n)/(N-1))) \qquad (n = 0, 1, ..., N-1)$$

$$w(n) = h(n) \Big( e^{-\frac{2\pi i}{N}(-2(n-((N-1)/2)))} + e^{-\frac{2\pi i}{N}(-1(n-((N-1)/2)))} + e^{-\frac{2\pi i}{N}(0(n-((N-1)/2)))} + e^{-\frac{2\pi i}{N}(1(n-((N-1)/2)))} + e^{-\frac{2\pi i}{N}(2(n-((N-1)/2)))} \Big) \qquad (n = 0, 1, ..., N-1)$$

2) weight the input vector $\bar{x}$ with the WOLA window $\bar{w}$:

$$y(n) = x(n)w(n) \qquad (n = 0, 1, ..., N-1)$$

3) overlap and add the weighted input vector $\bar{y}$:

$$\begin{array}{lllllllllll}
z(0) & = & y(0) & + & y(8) & + & y(16) & + & y(24) & + & y(32) \\
z(1) & = & y(1) & + & y(9) & + & y(17) & + & y(25) & + & y(33) \\
z(2) & = & y(2) & + & y(10) & + & y(18) & + & y(26) & + & y(34) \\
z(3) & = & y(3) & + & y(11) & + & y(19) & + & y(27) & + & y(35) \\
z(4) & = & y(4) & + & y(12) & + & y(20) & + & y(28) & + & y(36) \\
z(5) & = & y(5) & + & y(13) & + & y(21) & + & y(29) & + & y(37) \\
z(6) & = & y(6) & + & y(14) & + & y(22) & + & y(30) & + & y(38) \\
z(7) & = & y(7) & + & y(15) & + & y(23) & + & y(31) & + & y(39)
\end{array}$$

4) transform the shorter weighted and overlapped vector $\bar{z}$:

$$\bar{X} = \text{FFT8}(\bar{z})$$

5) for critical sampling, shift eight input vector samples (effectively: $x(0) = x(8), x(1) = x(9), ..., x(39) = x(47)$) and then return to step 2.

The example eight-channel, five-overlap WOLA FFT hopefully is sufficiently clear such that it can easily be generalized to a WOLA FFT of any length and number of overlaps.